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D-instanton derivation of multi-fermion F-terms in supersymmetric QCD

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Abstract: We investigate effects of field theory instantons by considering D-instantons in a suitable D3-brane background. In supersymmetric QCD with SU(N_c) gauge group and $N_f = N_c$ flavors, the moduli space of vacua is deformed by instantons. This effect can be described by the chiral interactions which are called multi-fermion F-terms. We derive these chiral interaction terms as D-instanton effects in the presence of D3-branes. For SU(2), the obtained result agrees with the previous result worked out by Beasley and Witten [1]. We also explicitly work out those for the case of the symplectic gauge group, and show that they describe the deformation of the moduli space.

Keywords: Supersymmetric gauge theory, Brane Dynamics in Gauge Theories, D-branes, Nonperturbative Effects.

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1. Introduction

Instanton physics in string theory is an interesting area to study. There are various non-perturbative objects such as D-branes, membranes and 5-branes and one can consider the instanton effects arising from such objects wrapping on some suitable cycles [2 − 4]. Space-time approach or the physical gauge approach was initiated by [5] and various interesting works were done. Since some field theories can be embedded in a string theory, field theory instantons can be understood in terms of instantons in string theory and this approach shed much light on the understanding of the structures of gauge theory instantons, such problem as the measure of the instanton moduli space. One of the important discovery is the D5-brane as a small instanton in Type I theory [2] and this led to the much progress in our understanding of the field theory instantons. Also in the string theory set up, one can have truly stringy instantons in the embedded field theory [6 − 11], whose effect cannot be reproduced within field theory and this leads to many interesting physics such as dynamical supersymmetry breaking combined with the other effects.

In this work we will use the string theory setup to understand one aspect of gauge theory instanton [12]. In particular we are interested in $\mathcal{N} = 1$ supersymmetric QCD and derive one interesting effect coming from the gauge theory instanton. We realize the gauge theory as a suitable D3-brane configuration [13] where D-instanton plays the role of the usual gauge theory instanton. In $\mathcal{N}=1$ SU(N_c) gauge theory with N_f fundamental flavors, the well known instanton effect is the generation of the ADS superpotential [14] for $N_c = N_f - 1$, which lifts the moduli of vacua (for a review, see [15]). This superpotential can be reproduced by the D-brane effective theory [16]. Various computation of the superpoetntial using D-instanton method can be found in [17, 18] and references in [19]. The ADS superpotential in D-brane theory with orientifolding can be found in [20]. For $N_f = N_c$, no superpotential is generated but a quantum effect deforms the complex structure of the moduli space [21]. In [1], it was pointed out that the deformation of the moduli space is still due to the usual gauge theory instanton, which gives rise to a chiral interaction. This interaction is a four-fermion interaction term and called multi-fermion F-terms. For $N_f > N_c$, instantons generate interaction terms with more fermions.

In this paper, we reproduce these multi-fermion F-terms from the D-brane effective theory, which is realized as D3-branes at a particular orbifold singularity. Though only the scalars which parametrize the moduli of $\mathcal{N}=1$ SQCD are involved in the calculation of the ADS superpotential, we have to include interactions of fermions in the quark superfields. These F-terms are given in the form of the integral with respect to the moduli of the instanton. It is difficult to perform the integration for general case. Hence, we calculate the simplest case $N_c = N_f = 2$. In this case, we can easily perform the integration and obtain the multi-fermion F-terms which are equivalent to the result of [1]. We study the case of the symplectic group as another example, and make a connection with the deformation of the moduli space. In this case, we can also explicitly carry out the integration, since the ADHM constraint is absent and the structure of the instanton moduli is simple.

This paper is organized as follows. In section 2, we briefly recall the basic setup of the D-brane effective theory which describes instantons in the $\mathcal{N}=4$ super Yang-Mills [12]. In section 3, we introduce D3-branes on one particular orbifold and realize the $\mathcal{N}=1$ SQCD with SU(N_c) gauge group [13]. After that, we discuss basic facts about the derivation of multi-fermion F-terms and show the general expression of multi-fermion F-terms for $N_f = N_c$. In section 4, we briefly review the multi-fermion F-terms and their relation to the deformation of the moduli space [1]. In section 5, we describe details of calculations in the simplest case of $N_f = N_c = 2$. We show that our result coincides with that in [1]. In section 6, we consider the more general case of $USp(N_c)$. We derive an expression of multi-fermion F-terms and show that they describe the deformation of the moduli space for $N_f = N_c + 1$. We also consider multi-fermion F-terms with more fermions, which appear for $N_f > N_c + 1$. section 7 is devoted for conclusions and discussions.

2. Preliminaries

In this section, we explain the basic setup of the system. It is well known that the D-instanton is the gauge theory instanton if the gauge theory in consideration is realized by D3-brane configurations [12]. Here we describe the massless spectrum of various sectors

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coming from D3 branes and D-instantons in the flat space and specify the interaction terms between various sectors. We divide open string fields into the following three sectors.

D3-D3 sector. This sector consists of the open strings whose both ends are on the D3-branes. At low energy, only massless modes of these open strings contribute to the theory. These massless fields form $\mathcal{N}=4$ super Yang-Mills multiplets. Let x^μ denote the world-volume coordinates of D3-branes and x^a denote the transverse coordinates of the remaining six dimensions. The bosonic components are denoted by A^μ and X^a, and their fermionic partners by Ψ^A_α and $\bar{\Psi}^{\dot{\alpha}}_A$. Indices of A are those of SU(4) denoting the chirality of the transverse six dimensions while α and $\dot{\alpha}$ denote the usual four dimensional chirality. We also write six scalars X^a in the antisymmetric representation

$$\Phi_{AB} \equiv \bar{\Sigma}^a_{AB} X^a. \tag{2.1}$$

where $\bar{\Sigma}^a$ (and Σ^a) realize six-dimensional Clifford algebras, and appear in the expression of six-dimensional γ-matrices,

$$\Gamma^a = \begin{pmatrix} 0 & \Sigma^a \\ \bar{\Sigma}^a & 0 \end{pmatrix}. \tag{2.2}$$

When we put N D3-branes, all of theses fields are in the adjoint representation of SU(N), and can be written as $N \times N$ hermitian matrices.

D(-1)-D(-1) sector. This sector consists of the open strings with both ends on the D-instantons. These are obtained by the dimensional reduction of ten-dimensional super Yang-Mills theory. The bosonic fields are written as a^μ, χ^a and the fermionic modes are denoted as $M^{\alpha A}, \lambda_{\dot{\alpha}A}$. Here we have adopted the ADHM inspired notation. We also introduce the triplet of the auxiliary fields D^c. These are expressed in terms of $k \times k$ matrices for a background with k D-instantons. It is argued in [12] that there are subtleties to obtain the D-instanton action in the presence of D3-branes by taking $\alpha' \to 0$ limit. This is because the coupling constant dependence on α' is different for D3-branes and D-instantons. If we take the coupling constant of D3-branes to be constant, the coupling constant of D-instanton diverges. In order to obtain the usual interacting theory for D(-1) massless modes, we need a suitable rescaling of the fields on D-instantons. Here we assume that such rescaling is already taken. We will refer to the rescaled fields by using the above notations.

D3-D(-1) sector. This sector includes the massless modes of the open strings stretching between the D3-branes and D-instantons. From the Neveu-Schwarz(NS) sector we have a bosonic spinor in the first four-directions where the GSO projection picks up the negative chirality. In the conjugate sector we obtain an independent bosonic spinor with the same chirality. We will refer to them as $\omega_{\dot{\alpha}}$ and $\bar{\omega}_{\dot{\alpha}}$, respectively. From the Ramond sector and its conjugate sector, we obtain a pair of fermions μ^A and $\bar{\mu}^A$. These fields are $N \times k$ and $k \times N$ matrices, respectively.

We are specifying the action of various sectors and the SUSY transformations are worked out in appendix A. Massless modes of the D(-1)-D(-1) and D3-D(-1) sectors correspond to the moduli of the gauge instanton which appear in the ADHM construction [22, 23]. When carrying out the path integral, one obtains the measure of the instanton moduli space naturally. The action is given by

$$
\begin{aligned}
S_1 = \operatorname{tr}\Big(& -[a_\mu, \chi^a]^2 + \chi^a \bar{\omega}_{\dot{\alpha}} \omega^{\dot{\alpha}} \chi^a + \frac{1}{2} (\bar{\Sigma^a})_{AB} \bar{\mu}^A \mu^B \chi_a - \frac{i}{4} (\bar{\Sigma^a})_{AB} M^{\alpha A} [\chi_a, M^B_\alpha] \\
& + i(\bar{\mu}^A \omega_{\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}} \mu^A + \sigma^\mu_{\beta\dot{\alpha}} [M^{\beta A}, a_\mu]) \lambda^{\dot{\alpha}}_A - i D^c (\bar{\omega}^{\dot{\alpha}} (\tau^c)^{\dot{\beta}}_{\dot{\alpha}} \omega_{\dot{\beta}} + i \bar{\eta^c}_{\mu\nu} [a^\mu, a^\nu]) \Big)
\end{aligned}
\tag{2.3}
$$

where $\eta, \bar{\eta}$ are the usual 't Hooft symbols and τ^c denote the usual Pauli matrices.

Including interaction terms with the scalars in $\mathcal{N} = 4$ super Yang-Mills theory, we can also obtain the instanton effective action with non-zero VEVs of the scalars, which is given by,

$$
S_2 = \operatorname{tr}\left(\frac{1}{8} \varepsilon^{ABCD} \bar{\omega}_{\dot{\alpha}} \Phi_{AB} \Phi_{CD} \omega^{\dot{\alpha}} + \frac{i}{2} \bar{\mu}^A \Phi_{AB} \mu^B + \frac{1}{4} \varepsilon^{ABCD} \bar{\omega}_{\dot{\alpha}} \chi_{AB} \Phi_{CD} \omega^{\dot{\alpha}} \right). \tag{2.4}
$$

In this paper, we also include contributions from the fermions in the D3-brane field theory. The interaction terms with the fermions (and gauge fields) are as follows:

$$
S_3 = \operatorname{tr}\left(i \bar{\omega}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}}_A \mu^A - i \bar{\mu}^A \bar{\Psi}_{\dot{\alpha} A} \omega^{\dot{\alpha}} + \frac{1}{2} \bar{\omega}_{\dot{\alpha}} \bar{\sigma}^{\mu\nu\dot{\alpha}}_{\dot{\beta}} F_{\mu\nu} \omega^{\dot{\beta}} \right). \tag{2.5}
$$

We will mainly consider nonzero fermionic background with no gauge field background.

3. Multi-fermion *F*-terms from D-instantons

In this paper, we study nonperturbative effects of instantons in an $\mathcal{N} = 1$ gauge theory. One way to obtain $\mathcal{N} = 1$ theory is the orbifolding with putting D3-branes at the orbifold singularity [13]. There is an issue whether the worldvolume quiver gauge theory of branes at singularities is $\prod_i \mathrm{SU}(N_i)$ or $\prod_i \mathrm{U}(N_i)$. We can decouple the overall $\mathrm{U}(1)$ factor under which no matter is charged. And the issue is about $\mathrm{U}(1)$ factors which have charged matters. As explained at [24], these $\mathrm{U}(1)$ factors are infrared free, and the resulting gauge theories realized in a non-compact Calabi-Yau should be regarded as $\prod_i \mathrm{SU}(N_i)$ since $\mathrm{U}(1)$ couplings vanish in the infrared limit.

One subtle issue is the presence of D-term of anomalous $\mathrm{U}(1)$ factors of $\prod_i \mathrm{U}(N_i)$. For such a $\mathrm{U}(1)$ we have Green-Schwarz anomaly cancellation mechanism, where the branes contain the necessary coupling between the $\mathrm{U}(1)$ gauge field and a suitable two-form field $B_{\mu\nu}$, which is dual to a scalar B in four-dimension [25]. The anomalous $\mathrm{U}(1)$ gauge fields A_μ get massive by Higgs mechanism, i.e., through the coupling of the form $(A_\mu - \partial_\mu B)^2$. For orbifold singularities, such two form fields are RR fields coming from the twisted sectors of the orbifold theory. Even if the $\mathrm{U}(1)$ fields get massive, their D-term equations should be imposed to obtain the correct moduli space [26] of field theory of $\prod_i \mathrm{SU}(N_i)$. Another scalar coming from NS-NS sector, say ϕ, which belongs to the same supermultiplet as $B_{\mu\nu}$, plays the role of Fayet-Iliopoulos term and the D-term of the anomalous $\mathrm{U}(1)$ gauge

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field can be always set to zero by turning on the expectation value of ϕ. Since the fields arising from twisted sectors are localized at the orbifold singularities, these are dynamical fields. Turning on their expectation value is possible in the non-compact setting while the expectation value of bulk fields are not dynamical. Thus by adjusting the value of ϕ, U(1) D-term constraint is satisfied but the moduli space of low energy $\prod_i \mathrm{SU}(N_i)$ theory is intact.

From now on we take a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. We also introduce fractional D3-branes which are D5-branes wrapped on collapsed cycles at the singularity. The gauge group of a general configuration is given by $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3) \times \mathrm{U}(N_4)$ in the ultra-violet limit. We will refer to D3-brane configurations as (N_1, N_2, N_3, N_4). The configuration we are interested in is given by $(N_c, N_f, 0, 0)$. For this configuration, one can make D-term of relative U(1)s vanishing by tuning the vacuum expectation value of suitable fields of twisted sectors [27]. The resulting low energy theory has $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ gauge theory which yields the $\mathcal{N} = 1$ SQCD with gauged flavor symmetry. We neglect gauging of flavor symmetry and treat the model as the usual $\mathcal{N} = 1$ SQCD. This is justified since the computation we will carry out does not depend on the gauge couplings so that we can make the gauge coupling of $\mathrm{SU}(N_f)$ arbitrarily smaller than that of $\mathrm{SU}(N_c)$. In practice, this implies that we write the final answer in terms of gauge invariant observables of $\mathrm{SU}(N_c)$, not in terms of gauge invariant variables $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$. Same approach is adopted in the computation of ADS superpotential using the D-instanton method [10, 16]. Configurations of D-instantons (and fractional D-instantons) are characterized by a similar fashion, (k_1, k_2, k_3, k_4). It is taken to be $(1, 0, 0, 0)$ in order to describe the one-instanton background.

In order to obtain the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, we introduce the following complex coordinates,

$$z^1 = x^4 + ix^5, \qquad z^2 = x^6 + ix^7, \qquad z^3 = x^8 + ix^9. \tag{3.1}$$

The orbifold is constructed from the following two projections

$$g_1 : \qquad z^2 = -z^2 \quad \text{and} \quad z^3 = -z^3, \tag{3.2a}$$
$$g_2 : \qquad z^1 = -z^1 \quad \text{and} \quad z^3 = -z^3. \tag{3.2b}$$

The low energy effective theory of the D3-brane becomes the $\mathcal{N} = 1$ quiver gauge theory with bi-fundamental matters. This theory can be obtained by imposing an orbifold projection on the $\mathcal{N} = 4$ SYM. The orbifold projection acts on the bosonic fields in the D3-D3 sector as follows:

$$A_\mu = \gamma(g_i) A_\mu \gamma(g_i)^{-1}, \qquad \Phi_{AB} = \pm\gamma(g_i) \Phi_{AB} \gamma(g_i)^{-1}, \;\; i = 1, 2 \tag{3.3}$$

where the $\pm$ sign must be chosen according to the projections (3.2). Since the orbifold is abelian, the representation matrix $\gamma(g_i)$ can be diagonalized and written in terms of the

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block diagonal matrices:

$$\gamma(g_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma(g_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.4}$$

where size of each blocks is determined by the D3-brane and D-instanton configurations. They have the size of (N_1, N_2, N_3, N_4) for the Chan-Paton factors on D3-branes, and (k_1, k_2, k_3, k_4) for those on D-instantons.

Now, we take the configuration of $\mathcal{N} = 1$ SQCD, namely, $(N_c, N_f, 0, 0)$ for D3-branes and $(1, 0, 0, 0)$ for D-instantons. The fields which survive in the orbifolding are as follows. First, we consider fields on the D3-branes. We have the gauge group $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ and regard the former as a color gauge group and the latter as a flavor group. Among the scalar fields, only two real (one complex) scalars have non-zero components. By using SU(4) notation of R-symmetry, these scalars are Φ_{23} and Φ_{14}, and they are hermitian conjugate to each other. Non-zero components are

$$\Phi_{23} = \begin{pmatrix} 0 & Q & 0 & 0 \\ \widetilde{Q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Phi_{14} = \begin{pmatrix} 0 & \widetilde{Q}^\dagger & 0 & 0 \\ Q^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

These two scalars are bi-fundamental in $(N_c, \bar{N}_f)$ and its conjugate. Concerning fermions, two fermions Ψ^1 and Ψ^4, (and their hermitian conjugates) survive. The fermion Ψ^1 has non-zero components in a similar fashion to the scalar fields:

$$\Psi^1 = \begin{pmatrix} 0 & \psi & 0 & 0 \\ \widetilde{\psi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \bar{\Psi}_1 = \begin{pmatrix} 0 & \bar{\widetilde{\psi}} & 0 & 0 \\ \bar{\psi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

Fermions ψ and $\widetilde{\psi}$ are the superpartners of the scalars Q and $\widetilde{Q}$ respectively. The other surviving fermion Ψ^4 becomes the gaugino $\lambda^{(\mathrm{g})}$ and has two non-zero block-diagonal components.

For the fields in the D(-1)-D(-1) sector, the situation is similar, but much more components vanish. Since we set all blocks to be zero except for the first, only the (first) diagonal block survives, and χ^a's are projected out. Furthermore, a_μ and $M^{\alpha A}$ do not appear in the action. Then, relevant fields in this sector are $\lambda^{\dot\alpha} \equiv \lambda^{\dot\alpha}_4$ and D^c.[1] In the D3-D(-1) sector, the first diagonal components of $\omega_{\dot\alpha}$, $\bar\omega_{\dot\alpha}$, $\mu \equiv \mu^4$, $\bar\mu \equiv \bar\mu^4$ survive, and are in the fundamental and anti-fundamental representations of $\mathrm{SU}(N_c)$, respectively. Off-diagonal components of $\mu' \equiv \mu^1$ and $\bar\mu' \equiv \bar\mu^1$ also survive and they are in the fundamental and anti-fundamental representations of $\mathrm{SU}(N_f)$, respectively.

[1] We also use the notation of λ, μ and μ' for their surviving components.

After the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold projection, the action is obtained from the action of D3 and D(-1) branes after keeping terms compatible with the orbifold projections. The part containing only the fields in D(-1)-D(-1) and D3-D(-1) sectors is greatly simplified and turns out to be

$$S_1 = i(\bar{\mu}_u \omega^u_{\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u}\mu^u)\lambda^{\dot{\alpha}} - iD_c \bar{\omega}^{\dot{\alpha}}_u (\tau^c)_{\dot{\alpha}}{}^{\dot{\beta}} \omega^u_{\dot{\beta}}. \tag{3.7}$$

The interaction terms with scalars are

$$S_2 = \frac{1}{2}\bar{\omega}_{\dot{\alpha}\,u}(Q^u_f Q^{\dagger f}_v + \widetilde{Q}^{\dagger u}_f \widetilde{Q}^f_v)\omega^{\dot{\alpha}v} - \frac{i}{2}\bar{\mu}_u \widetilde{Q}^{\dagger\,u}_f \mu'^f + \frac{i}{2}\bar{\mu}'_f Q^{\dagger\,f}_u \mu^u. \tag{3.8}$$

The above action is equivalent to those appearing in the ADHM construction. Here, we also include the interaction terms with fermions. These terms and the interactions with gauge fields and gauginos are

$$\begin{aligned} S_3 = &- i\bar{\mu}'_f \bar{\psi}^f_{\dot{\alpha}u}\omega^{\dot{\alpha}u} + i\bar{\omega}_{\dot{\alpha}\,u}\bar{\widetilde{\psi}}_f^{\dot{\alpha}\,u}\mu'^f - i\bar{\mu}_u \bar{\lambda}^{(\mathrm{g})}_{\dot{\alpha}v}{}^{u}\omega^{\dot{\alpha}v} \\ &+ i\bar{\omega}_{\dot{\alpha}\,u}\bar{\lambda}^{(\mathrm{g})}_v{}^{\dot{\alpha}u}\mu^v + \frac{1}{2}\bar{\omega}_{\dot{\alpha}\,u}\bar{\sigma}^{\mu\nu\dot{\alpha}}_{\dot{\beta}}(F_{\mu\nu})^u_v\omega^{\dot{\beta}\,v} \end{aligned} \tag{3.9}$$

In this paper, we mainly investigate the contribution from the fermionic components of quarks, ψ and $\widetilde{\psi}$. We do not consider the terms with the gauge fields and gauginos, and S_3 indicates only terms with $\bar{\psi}$ or $\bar{\widetilde{\psi}}$, hereafter.

The space-time approach to obtain the F-term contribution from D-instanton was explained in [28]. Following that, the term L_D induced by one D-instanton is given by the following path integral

$$L_D = \int d\{a, \chi, M, \lambda, D, \omega, \bar{\omega}, \mu, \bar{\mu}\}\; e^{-S_1 - S_2 - S_3}. \tag{3.10}$$

We are taking a suitable $\alpha' \to 0$ limit for D3/D(-1) system as explained in section 2. In this limit, massless modes of D(-1) brane theory are described by zero-dimensional field theory and the path integral is reduced to the usual integral. The space-time approach applied to D3/D(-1) system in the $\alpha' \to 0$ limit reproduces the gauge theory instanton dynamics.[2] In the (path) integral evaluation, a^μ gives rise to four bosonic translation modes x^μ which represent the position of the instanton. The upper left component of $M^{\alpha 4}$ gives rise to super-translation modes θ^α, since it is superpartner of a_μ. Then, the F-term contributions from D-instantons can be expressed as

$$L_D = \int d^4x\, d^2\theta\; W, \tag{3.11}$$

where W should be calculated by performing path-integral with the above bosonic and fermionic modes suppressed:

$$W = \int d\{\lambda, D, \omega, \bar{\omega}, \mu, \bar{\mu}\}\; e^{-S_1 - S_2 - S_3}. \tag{3.12}$$

[2]There is ambiguity in the overall constant factor since the partition function is defined up to the overall constant factor. We will omit the overall normalization constant hereafter.

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The integrals over D and λ enforce the bosonic and fermionic ADHM constraints. If we carry out these integrations we have

$$W = \int d\{\omega, \bar{\omega}, \mu, \bar{\mu}\} \delta(\bar{\mu}_u \omega^u_{\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}\,u} \mu^u) \delta(\bar{\omega}^{\dot{\alpha}}_u (\tau^c)_{\dot{\alpha}}^{\ \dot{\beta}} \omega^u_{\dot{\beta}}) e^{-S_2 - S_3}. \tag{3.13}$$

In the computation of the ADS superpotential, one can consider only the contribution from S_2. If we consider the SU(N_c) gauge theory, one can easily see that $N_f = N_c - 1$ is the condition for nonvanishing fermionic integration from the structure of S_2. From S_2 one pair of μ^u, μ'^f comes down to the integrand simultaneously and $\delta(\bar{\mu}_v \omega^u_{\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}\,u} \mu^v)$ will give additional μ^u to the integration. Thus we need $N_f = N_c - 1$ for the nonzero fermionic integration. As shown in [10, 16] the explicit integration gives the usual ADS superpotential. Now we consider the $N_f = N_c$ case. For nonzero fermionic integration, we need S_3 to pull down two more fermion terms after pulling down fermion terms up to $N_c - 1$ from S_2. After carrying out these integrations, we obtain the following expression:

$$\begin{aligned} W = \int & d^2\omega\, d^2\bar{\omega}\ \delta^{(3)}(\bar{\omega}_{\dot{\alpha}u} (\tau^c)^{\dot{\alpha}}_{\dot{\beta}} \omega^{\dot{\beta}u})\, e^{-\frac{1}{2}\bar{\omega}_{\dot{\alpha}u}\left(Q^u_f Q^{\dagger f}_v + \widetilde{Q}^{\dagger u}_f \widetilde{Q}^f_v\right)\omega^{\dot{\alpha}v}} \\ & \times \Bigg\{ \frac{1}{2N!(N-2)!} \Big[\varepsilon_{uvt_1\cdots t_{N-2}} \varepsilon^{fgh_1\cdots h_{N-2}} \varepsilon^{s_1\cdots s_N} \varepsilon_{k_1\cdots k_N} \\ & \qquad \times \omega^u_{\dot{\alpha}} \omega^{\dot{\alpha}v} \bar{\omega}_{\dot{\beta}x} \bar{\tilde{\psi}}^{\dot{\beta}x}_f\, \bar{\omega}_{\dot{\gamma}y} \bar{\tilde{\psi}}^{\dot{\gamma}y}_g\, \widetilde{Q}^{\dagger t_1}_{h_1} \cdots \widetilde{Q}^{\dagger t_{N-2}}_{h_{N-2}} Q^{\dagger k_1}_{s_1} \cdots Q^{\dagger k_N}_{s_N} \Big] \\ & - \frac{2}{(N-1)!(N-1)!} \Big[\varepsilon_{ut_1\cdots t_{N-1}} \varepsilon^{fh_1\cdots h_{N-1}} \varepsilon^{vs_1\cdots s_{N-1}} \varepsilon_{gk_1\cdots k_{N-1}} \\ & \qquad \times \omega^u_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}}_v \bar{\omega}_{\dot{\beta}x} \bar{\tilde{\psi}}^{\dot{\beta}x}_f\, \bar{\psi}^g_{\dot{\gamma}y} \omega^{\dot{\gamma}y} \widetilde{Q}^{\dagger t_1}_{h_1} \cdots \widetilde{Q}^{\dagger t_{N-1}}_{h_{N-1}} Q^{\dagger k_1}_{s_1} \cdots Q^{\dagger k_{N-1}}_{s_{N-1}} \Big] \\ & + \frac{1}{2N!(N-2)!} \Big[\varepsilon_{t_1\cdots t_N} \varepsilon^{h_1\cdots h_N} \varepsilon^{uvs_1\cdots s_{N-2}} \varepsilon_{fgk_1\cdots k_{N-2}} \\ & \qquad \times \omega^u_{\dot{\alpha}} \omega^{\dot{\alpha}v} \bar{\omega}_{\dot{\beta}x} \bar{\tilde{\psi}}^{\dot{\beta}x}_f\, \bar{\omega}_{\dot{\gamma}y} \bar{\tilde{\psi}}^{\dot{\gamma}y}_g\, \widetilde{Q}^{\dagger t_1}_{h_1} \cdots \widetilde{Q}^{\dagger t_N}_{h_N} Q^{\dagger k_1}_{s_1} \cdots Q^{\dagger k_{N-2}}_{s_{N-2}} \Big] \Bigg\}, \end{aligned} \tag{3.14}$$

where $N = N_f = N_c$. As in the case of the ADS superpotential, this expression represents lowest components of the effective superpotential. The fermions $\bar{\psi}_{\dot{\alpha}}$ and $\bar{\tilde{\psi}}_{\dot{\alpha}}$ can be regarded as the lowest components of the superfields $\bar{D}_{\dot{\alpha}} \bar{Q}$ and $\bar{D}_{\dot{\alpha}} \bar{\tilde{Q}}$, where Q's are promoted to the superfields. Schematically, this is in the form of

$$\bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} F(Q, Q^\dagger) \to \bar{D}_{\dot{\alpha}} \bar{Q} \bar{D}^{\dot{\alpha}} \bar{Q} F(Q, \bar{Q}). \tag{3.15}$$

By this computation we obtain terms in the effective action, which involve two derivatives of bosonic fields or four fermions. This term is not manifestly supersymmetric since it contains non-holomorphic terms. However, in [1] it is shown that (3.15) is chiral in the on-shell supersymmetry algebra because it is related to a representative of a Dolbeault cohomology whose elements parametrize infinitesimal deformations of moduli space [1].[3]

[3] This type of the effective potential was also considered in [29].

Similarly one can clearly see that the nonvanishing contribution can be obtained even for $N_f > N_c$. We just have to pull down the necessary terms from S_3. These terms are called multi fermion F-terms since these represent interactions involving many fermions. Note that in the space-time approach, the derivation of the usual superpotential F-term and the multi-fermion F-term is uniform. We just have to work out the path integral to figure out the contribution to the effective action and the zero mode structure determines whether the resulting contribution of D-instanton is the superpotential or multi-fermion F-terms. We denote the both contribution as W. Whether W means the usual superpotential or multi-fermion F-term should be clear depending on the context.

It is a formidable task to perform integration in general cases, and we consider the simplest cases. We will consider the case of SU(2) with $N_f = N_c = 2$. In this case, multi-fermion F-term gives rise to the moduli deformation. In the next section, we review the relation between multi-fermion F-terms and the moduli space deformation.

4. Review on multi-fermion F-terms

In this section, we briefly review the multi-fermion F-terms. The simplest multi-fermion F-term has a four-fermion interaction. In this case, the multi-fermion F-term can be connected to the deformation of the moduli space of vacua.

Let us consider a model whose moduli space is classically described by the complex fields ϕ^I satisfying a holomorphic equation,

$$\mathcal{C}(\phi) = 0. \tag{4.1}$$

The effective action is written in terms of ϕ as

$$S = \int d^4x\, d^4\theta\, K(\Phi, \bar{\Phi}) \tag{4.2}$$

where Φ^I and $\bar{\Phi}^{\bar{I}}$ are chiral and anti-chiral superfields whose lowest components are ϕ^I and $\bar{\phi}^{\bar{I}}$, respectively. Then, the bosonic part of this action is

$$S = \int d^4x\, g_{I\bar{J}} \partial\phi^I \partial\bar{\phi}^{\bar{J}}, \tag{4.3}$$

where $g_{I\bar{J}}$ is metric on the moduli space, and obtained from the Kähler potential as

$$g_{I\bar{J}} = \frac{\partial^2 K(\phi,\bar{\phi})}{\partial\phi^I \partial\bar{\phi}^{\bar{J}}}. \tag{4.4}$$

Suppose that the governing equation on the moduli space is deformed to

$$\mathcal{C}(\phi) = \epsilon. \tag{4.5}$$

This deformation of the complex structure can be represented as the change of the basis of holomorphic one forms

$$d\phi^I \to d\phi^I - \omega^I_{\bar{J}} d\bar{\phi}^{\bar{J}}. \tag{4.6}$$

Then, the metric on the moduli space changes as

$$g_{I\bar{J}} d\phi^I d\bar{\phi}^{\bar{J}} \to g_{I\bar{J}} \left(d\phi^I - \omega^I_{\bar{K}} d\bar{\phi}^{\bar{K}} \right) d\bar{\phi}^{\bar{J}}, \tag{4.7}$$

and the sigma model action receives the correction term

$$\delta S = \int d^4x \, d^2\theta \, \omega_{\bar{I}\bar{J}} \bar{D}_{\dot{\alpha}} \bar{\Phi}^{\bar{I}} \bar{D}^{\dot{\alpha}} \bar{\Phi}^{\bar{J}} = \int d^4x \, \omega_{\bar{I}\bar{J}} \partial \bar{\phi}^{\bar{I}} \partial \bar{\phi}^{\bar{J}} + \cdots \tag{4.8}$$

$$\omega_{\bar{I}\bar{J}} = g_{K\bar{I}} \omega^K_{\bar{J}}. \tag{4.9}$$

Thus, the deformation of the moduli space is related to the multi-fermion F-terms. Note that away from the singularities on the moduli space, this deformation can be converted to the non-holomorphic change of the valuable:

$$\phi^I \to \widetilde{\phi}^I = \phi^I + \delta\phi^I(\phi, \bar{\phi}), \tag{4.10}$$

where $\widetilde{\phi}^I$ satisfies the classical constraint

$$\mathcal{C}(\widetilde{\phi}) = 0. \tag{4.11}$$

In the next section, we work out the SU(2) case and see that the result agrees with the computation carried out in [1].

5. Computation of multi-fermion F-terms

In the previous section 3, we have seen that multi-fermion F-terms in the form of the integral with respect to instanton moduli. However, it is difficult to evaluate the integral for general SU(N). In this section, we investigate the simplest case of multi-fermion F-terms, i.e. SU(N) SQCD for $N_f = N_c = 2$. In this case, multi-fermion F-terms are calculated in [1]. We show that our D-instanton derivation correctly reproduces their result, which is related to the moduli space deformation.

We start with the expression of (3.12),

$$W = \int d\{\lambda, D, \omega, \bar{\omega}, \mu, \bar{\mu}\} \; e^{-S_1 - S_2 - S_3}. \tag{5.1}$$

We first perform integrations with respect to fermions. Integrals over $\lambda_{\dot{\alpha}}$ enforce the fermionic ADHM constraints. Here, we do not consider the terms with gauginos (and gauge fields). By neglecting these terms, integrals over μ's determine terms which contribute to the multi-fermion F-term. After these integrations, we obtain the following expression of W, up to an overall constant:

$$W = \int d^2\omega \, d^2\bar{\omega} \, d^3D \; \widetilde{W} \, e^{-\frac{1}{2}\bar{\omega}(QQ^\dagger + \widetilde{Q}^\dagger \widetilde{Q})\omega + iD^c \bar{\omega}\tau^c \omega}, \tag{5.2}$$

$$\begin{aligned} \widetilde{W} = {} & \frac{1}{4} \varepsilon_{u_1 u_2} \varepsilon^{f_1 f_2} \omega^{u_1}_{\dot{\alpha}} \omega^{\dot{\alpha}\, u_2} \bar{\omega}_{\dot{\beta}\, x} \bar{\widetilde{\psi}}^{\dot{\beta}\, x}{}_{f_1} \bar{\omega}_{\dot{\gamma} y} \bar{\widetilde{\psi}}^{\dot{\gamma}\, y}{}_{f_2} \varepsilon^{v_1 v_2} \varepsilon_{g_1 g_2} Q^{\dagger\, g_1}_{v_1} Q^{\dagger\, g_2}_{v_2} \\ & + 2 \varepsilon_{u_1 u_2} \varepsilon^{f_1 f_2} \omega_{\dot{\alpha}}{}^{u_1} \widetilde{Q}^{\dagger}{}^{u_2}_{f_2} \bar{\omega}_{\dot{\beta}\, x} \bar{\widetilde{\psi}}^{\dot{\beta}\, x}{}_{f_1} \varepsilon^{v_1 v_2} \varepsilon_{g_1 g_2} \bar{\omega}^{\dot{\alpha}}{}_{v_1} \bar{\psi}^{\dot{\gamma}\, g_1}_{y} \omega_{\dot{\gamma}}{}^{y} Q^{\dagger\, g_2}_{v_2} \\ & + \frac{1}{4} \varepsilon_{v_1 v_2} \varepsilon^{g_1 g_2} \widetilde{Q}^{\dagger\, v_1}_{g_1} \widetilde{Q}^{\dagger\, v_2}_{g_2} \varepsilon^{u_1 u_2} \varepsilon_{f_1 f_2} \bar{\omega}_{\dot{\alpha}\, u_1} \bar{\omega}^{\dot{\alpha}}{}_{u_2} \bar{\psi}^{\dot{\beta}\, f_1}_{x} \omega_{\dot{\beta}}{}^{x} \bar{\psi}^{\dot{\gamma}\, f_2}_{y} \omega^{y}_{\dot{\gamma}}. \end{aligned} \tag{5.3}$$

Using the Fierz identity[4] for the first and the third terms, we obtain an expression with one pair of $\omega, \bar{\omega}$ with their spinor indices contracted, i.e.,

$$(\omega^u \omega^v)(\bar{\omega}_x \bar{\widetilde{\psi}}^x{}_f) = -(\omega^u \bar{\widetilde{\psi}}^x{}_f)(\omega^v \bar{\omega}_x) + (\omega^v \bar{\widetilde{\psi}}^x{}_f)(\omega^u \bar{\omega}_x), \tag{5.4}$$

where spinor indices are contracted in each parenthesis. Then, $\widetilde{W}$ can be expressed as

$$\begin{aligned}\widetilde{W} = \bar{\omega}_{\dot{\alpha}u_1}\omega^{\dot{\alpha}v_1}\bar{\omega}_{\dot{\beta}u_2}\omega^{v_2}_{\dot{\gamma}}\Bigg[&\frac{1}{2}\varepsilon_{v_1v_2}\varepsilon^{f_1f_2}\bar{\widetilde{\psi}}^{\dot{\beta}u_1}{}_{f_1}\bar{\widetilde{\psi}}^{\dot{\gamma}\,u_2}{}_{f_2}\varepsilon^{w_1w_2}\varepsilon_{g_1g_2}Q^{\dagger\,g_1}_{w_1}Q^{\dagger\,g_2}_{w_2} \\ &-2\varepsilon_{v_1w_1}\varepsilon^{f_1f_2}\widetilde{Q}^{\dagger\,w_1}{}_{f_2}\bar{\widetilde{\psi}}^{\dot{\beta}\,u_2}{}_{f_1}\varepsilon^{u_1w_2}\varepsilon_{g_1g_2}\bar{\psi}^{\dot{\gamma}\,g_1}{}_{v_2}Q^{\dagger\,g_2}{}_{w_2} \\ &+\frac{1}{2}\varepsilon_{w_1w_2}\varepsilon^{g_1g_2}\widetilde{Q}^{\dagger\,w_1}_{g_1}\widetilde{Q}^{\dagger\,w_2}_{g_2}\varepsilon^{u_1u_2}\varepsilon_{f_1f_2}\bar{\psi}^{\dot{\beta}\,f_1}{}_{v_1}\bar{\psi}^{\dot{\gamma}\,f_2}{}_{v_2}\Bigg].\end{aligned} \tag{5.5}$$

Next, we perform the integration with respect to ω and $\bar{\omega}$. After some algebras, we obtain the following relation (see appendix C):

$$\begin{aligned}F(A) &= \int d^2\omega\, d^2\bar{\omega}\, d^3D\; e^{-\bar{\omega}A\omega + iD^c\bar{\omega}\tau^c\omega} \\ &= \int d^3D \frac{1}{\det(A^2+D^2)} \\ &= (\mathrm{tr}\,A)^{-1}.\end{aligned} \tag{5.6}$$

where the matrix A is a 2×2 hermitian matrix, and given by $A = \frac{1}{2}(QQ^\dagger + \widetilde{Q}^\dagger\widetilde{Q})$ in the present case. Picking up relevant parts in (5.2) and (5.5), we obtain

$$\begin{aligned}&\int d\omega\, d\bar{\omega}\, d^3D\; \bar{\omega}_{\dot{\alpha}u}\omega^{\dot{\alpha}v}\bar{\omega}_{\dot{\beta}\,u'}\omega^{v'}_{\dot{\gamma}}\, e^{-\bar{\omega}A\omega + iD^c\bar{\omega}\tau^c\omega} \\ &\qquad = \frac{1}{2}\varepsilon_{\dot{\beta}\dot{\gamma}}\int d\omega\, d\bar{\omega}\, d^3D\; \bar{\omega}_{\dot{\alpha}u}\omega^{\dot{\alpha}v}\bar{\omega}_{\dot{\alpha}'u'}\omega^{\dot{\alpha}'v'}\, e^{-\bar{\omega}A\omega + iD^c\bar{\omega}\tau^c\omega} \\ &\qquad = \frac{1}{2}\varepsilon_{\dot{\beta}\dot{\gamma}}\frac{\partial}{\partial A^u_v}\frac{\partial}{\partial A^{u'}_{v'}}F(A) \\ &\qquad = \varepsilon_{\dot{\beta}\dot{\gamma}}\frac{\delta^v_u\delta^{v'}_{u'}}{(\mathrm{tr}\,A)^3}.\end{aligned} \tag{5.7}$$

Using this, the multi-fermion F-term is given by

$$\begin{aligned}W = (\mathrm{tr}\,A)^{-3}\Bigg(&-\frac{1}{2}\varepsilon^{uv}\varepsilon^{fg}\bar{\widetilde{\psi}}^u{}_{\dot{\alpha}\,f}\bar{\widetilde{\psi}}^{\dot{\alpha}\,v}{}_f\,\varepsilon^{u'v'}\varepsilon_{f'g'}Q^{f'}_{u'}Q^{g'}_{v'} \\ &+2\varepsilon_{ut}\varepsilon^{fk}\varepsilon^{us}\varepsilon_{gk}\bar{\widetilde{\psi}}^x{}_{\dot{\alpha}\,f}\bar{\psi}^{\dot{\alpha}\,g}_x\widetilde{Q}^{\dagger\,t}{}_k Q^{\dagger\,k}{}_s \\ &-\frac{1}{2}\varepsilon^{uv}\varepsilon^{fg}\widetilde{Q}^{\dagger\,u}_f\widetilde{Q}^{\dagger\,v}_f\varepsilon^{u'v'}\varepsilon_{f'g'}\bar{\psi}^{f'}_{\dot{\alpha}u'}\bar{\psi}^{\dot{\alpha}g'}_{v'}\Bigg).\end{aligned} \tag{5.8}$$

---

[4]For bosonic spinors A, B, C and D, we have the following relation:

$$A_{\dot{\alpha}}B^{\dot{\alpha}}C_{\dot{\beta}}D^{\dot{\beta}} = (AB)(CD) = -(AD)(BC) + (AC)(BD).$$

This can be used for color and flavor SU(2) indices as well.

In the case of SU(2), the fundamental and anti-fundamental representations coincide, and the N_f flavors can be treated as $2N_f$ flavors. Here, we treat the flavor symmetry as global symmetry by taking the limit where the gauge coupling constant of SU(N_f) is arbitrarily smaller than that of SU(N_c), which is possible since eq. (5.8) does not depend on the gauge coupling constants. Then, the flavor symmetry becomes SU(4). First, we rewrite the quarks $\widetilde{Q}$ in the fundamental representation of SU(2) color symmetry as

$$Q'^{uf} = \varepsilon^{uv}\widetilde{Q}^f_v, \qquad \psi'^{uf}_\alpha = \varepsilon^{uv}\widetilde{\psi}^f_{\alpha v}. \tag{5.9}$$

By applying the Fierz identity to color SU(2) indices, each term in the parenthesis becomes

$$-\varepsilon^{fg}\varepsilon_{f'g'}(Q'^{\dagger f'}\bar{\psi}_f)(Q'^{\dagger g'}\bar{\psi}_g), \tag{5.10}$$

$$2\varepsilon^{fg}\varepsilon_{f'g'}((\bar{\psi}_f Q^\dagger_g)(\bar{\psi}'^{f'}Q'^{\dagger g'}) + (\bar{\psi}_f Q'^{\dagger f'})(\bar{\psi}'^{g'}Q^\dagger_g)), \tag{5.11}$$

$$-\varepsilon^{fg}\varepsilon_{f'g'}(Q'^{\dagger f'}\bar{\psi}_f)(Q'^{\dagger g'}\bar{\psi}_g), \tag{5.12}$$

respectively, where SU(2) color indices are contracted inside each parenthesis and are not written explicitly. Thus, the multi-fermion F-term is given by

$$\begin{aligned} W = (\mathrm{tr}\, A)^{-3}\varepsilon^{fg}\varepsilon_{f'g'}\Big[&-(Q'^{\dagger f'}\bar{\psi}_f)(Q'^{\dagger g'}\bar{\psi}_g) \\ &+ 2\left((\bar{\psi}_f Q^\dagger_g)(\bar{\psi}'^{f'}Q'^{\dagger g'}) + (\bar{\psi}_f Q'^{\dagger f'})(\bar{\psi}'^{g'}Q^\dagger_g)\right) \\ &- (Q'^{\dagger f'}\bar{\psi}_f)(Q'^{\dagger g'}\bar{\psi}_g)\Big]. \end{aligned} \tag{5.13}$$

By promoting the quarks Q and their superpartners ψ to the superfields, fermion $\bar{\psi}_{\dot{\alpha}}$ can be interpreted as the lowest component of $\bar{D}_{\dot{\alpha}}\bar{Q}$. The quarks can be combined into $\mathcal{Q}$ as

$$\mathcal{Q}^u_i = \begin{cases} Q^u_f, & (i = 1, 2) \\ Q'^{uf}. & (i = 3, 4) \end{cases} \tag{5.14}$$

Then, the multi-fermion F-term can be expressed in the following compact form

$$W = (\mathrm{tr}\, A)^{-3}\varepsilon_{ijkl}\bar{D}_{\dot{\alpha}}(\varepsilon^{uv}\bar{\mathcal{Q}}^i_u\bar{\mathcal{Q}}^j_v)\bar{D}^{\dot{\alpha}}(\varepsilon^{u'v'}\bar{\mathcal{Q}}^k_{u'}\bar{\mathcal{Q}}^l_{v'}), \tag{5.15}$$

where the matrix A can be written as $A = \mathcal{Q}\bar{\mathcal{Q}}$. In the moduli space of SQCD, we have the D-flatness condition, $D^a = \mathrm{tr}\, \mathcal{Q}^\dagger\tau^a\mathcal{Q} = 0$, and we can rewrite $\mathrm{tr}\, A$ as

$$\mathrm{tr}\, A = \sqrt{\det \bar{\mathcal{Q}}\mathcal{Q}}. \tag{5.16}$$

Introducing the "meson" field $M_{ij} = \varepsilon_{uv}\mathcal{Q}^u_i\mathcal{Q}^v_j$, we obtain the following expression for the multi-fermion F-term

$$W = (\mathrm{tr}\, \bar{M}M)^{-\frac{3}{2}}\varepsilon^{ijkl}\bar{D}_{\dot{\alpha}}\bar{M}_{ij}\bar{D}^{\dot{\alpha}}\bar{M}_{kl}. \tag{5.17}$$

In [1], the multi-fermion F-term is derived by the usual field theory instanton calculation and its relation to the moduli space deformation is explained. In the case of SU(2), the classical moduli space is described by the meson M satisfying the classical constraint

$$\mathrm{Pf}\, M = 0. \tag{5.18}$$

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This constraint is modified by the quantum effect to

$$\mathrm{Pf}\, M = \epsilon. \tag{5.19}$$

This quantum moduli space can be converted to the classical moduli space by a non-holomorphic change of variables. Introducing the new coordinate $\widetilde{M} = M - \delta M$, the deformation of the complex structure gives rise to the additional term of the form

$$\omega_{ijkl} G^{ij}_{mn} \bar{D}\bar{M}^{kl} \bar{D}\bar{M}^{mn}, \tag{5.20}$$

where G^{ij}_{kl} is the metric of the moduli space, and ω_{ijkl} is given by

$$\left(\frac{\partial}{\partial \bar{M}^{kl}} + \omega_{ijkl} \frac{\partial}{\partial M_{ij}} \right) \widetilde{M}_{mn} = 0. \tag{5.21}$$

In this case, the new coordinate is given by

$$M_{ij} \to M_{ij} - \frac{\epsilon}{2} \frac{\varepsilon_{ijkl} \bar{M}^{kl}}{(\mathrm{tr}\, \bar{M}M)}. \tag{5.22}$$

Then, ω_{ijkl} becomes

$$\omega_{ijkl} = \frac{\epsilon}{2} \left(\frac{\varepsilon_{ijkl}}{(\mathrm{tr}\, \bar{M}M)} - \frac{\varepsilon_{ijmn} \bar{M}^{mn} M_{kl}}{(\mathrm{tr}\, \bar{M}M)^2} \right). \tag{5.23}$$

The metric can be determined by the asymptotic form of the Kähler potential $K = \sqrt{\mathrm{tr}\, \bar{M}M}$. The additional factor of $(\mathrm{tr}\, \bar{M}M)^{-1/2}$ comes from this metric.[5] Then, the multi-fermion F-term becomes,

$$W = (\mathrm{tr}\, \bar{M}M)^{-\frac{3}{2}} \left(\varepsilon_{ijkl} - \frac{\varepsilon_{ijmn} \bar{M}^{mn} M_{kl}}{(\mathrm{tr}\, \bar{M}M)} - \frac{\varepsilon_{mnkl} \bar{M}^{mn} M_{ij}}{(\mathrm{tr}\, \bar{M}M)} \right) \bar{D}\bar{M}^{ij} \bar{D}\bar{M}^{kl}, \tag{5.24}$$

where the third term has been added so that the multi-fermion F-term is manifestly symmetric to the exchange of two $\bar{D}\bar{M}$'s. It can be done because the second and third term vanish on the moduli space since

$$\varepsilon^{ijkl} \bar{M}_{kl} d\bar{M}_{ij} = d(\varepsilon^{ijkl} \bar{M}_{ij} \bar{M}_{kl}). \tag{5.25}$$

Using this relation (5.25) again, the multi-fermion F-term (5.17) is equivalent to (5.24) which is based on the moduli space deformation.

6. Case of symplectic gauge group

In this section, we consider the case of the symplectic gauge group, which can be obtained by introducing the orientifold [30]. Here, we take the orbifolding with $(2N_c, 2N_f, 0, 0)$ for D3-branes and $(1, 0, 0, 0)$ for D-instantons, and put the O3-plane on the orbifold singularity.

[5] There are another additional terms. We do not write these terms explicitly because they vanish on the moduli space.

Then, the color and flavor gauge groups become $USp(2N_c)$ and $USp(2N_f)$, respectively. If one neglects the flavor gauge symmetry and treats this as a global symmetry, this remains $\mathrm{U}(2N_f)$.

The orientifold acts on the Chan-Paton factor as a matrix $\gamma(\Omega)$. This matrix can be different for different D-branes, and have opposite symmetries (symmetric or anti-symmetric) for the D3-brane and D-instanton [31]. We take the anti-symmetric one for the D3-brane to obtain the symplectic gauge group. The matrix $\gamma(\Omega)$ must satisfy the following consistency condition:

$$\gamma(g)\gamma(\Omega)\gamma(g)^{\mathrm{T}} = \gamma(\Omega). \tag{6.1}$$

Then, the action of the orientifold is

$$\gamma_- = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & \widetilde{J} & 0 & 0 \\ 0 & 0 & J^{(3)} & 0 \\ 0 & 0 & 0 & J^{(4)} \end{pmatrix}, \tag{6.2}$$

where, J's are anti-symmetric matrices which satisfies $J^2 = -1$. We use the notation of $J_{uv} = -J^{uv} = (J^{-1})_{uv}$, and similarly for $\widetilde{J}$.

The orientifold imposes the following additional conditions on massless fields in the D3-D3 sector:

$$A_\mu = -\gamma_- A_\mu^{\mathrm{T}} \gamma_-^{-1}, \tag{6.3a}$$

$$\Phi = -\gamma_- \Phi^{\mathrm{T}} \gamma_-^{-1}, \qquad \Phi^\dagger = -\gamma_- \Phi^{\dagger\,\mathrm{T}} \gamma_-^{-1}, \tag{6.3b}$$

$$\Psi^A = -R^A_B \gamma_- (\Psi^B)^{\mathrm{T}} \gamma_-^{-1}, \qquad \bar{\Psi}_A = -\gamma_- (\bar{\Psi}_B)^{\mathrm{T}} \gamma_-^{-1} R^B_A, \tag{6.3c}$$

where the minus sign for the scalars is due to the spacetime reflection, and those for the gauge fields and the fermions come from the worldsheet reflection. The matrix R^A_B is the action of the reflection on six dimensional spinors:

$$R = -i\Gamma^{456789}. \tag{6.4}$$

The projection on the gauge fields (and gauginos) restricts the color and flavor gauge groups to the symplectic gauge group. For the chiral scalars and fermions, the conditions (6.3) make connections between Q and $\widetilde{Q}$, ψ and $\widetilde{\psi}$ as

$$\widetilde{Q}^f_u = -\widetilde{J}^{fg} Q^v_g J_{vu}, \quad \widetilde{Q}^{\dagger u}_f = -J^{uv} Q^{\dagger g}_v \widetilde{J}_{gf}, \quad \widetilde{\psi}^f_u = -\widetilde{J}^{fg} \psi^v_g J_{vu}, \quad \bar{\widetilde{\psi}}{}^u_f = -J^{uv} \bar{\psi}^g_v \widetilde{J}_{gf}. \tag{6.5}$$

Since these conditions do not give any restriction on the flavor symmetry, the global symmetry itself remains to be $\mathrm{U}(2N_f)$.

For the Chan-Paton factor on the D-instanton, the action of the orientifold is a symmetric matrix, which can be taken to be the unit matrix

$$\gamma_+ = 1. \tag{6.6}$$

For the fields in the D(-1)-D(-1) sector, the orientifold imposes the following conditions:

$$a_\mu = a_\mu^{\mathrm{T}}, \qquad \chi = -\chi^{\mathrm{T}}, \qquad M^A_\alpha = R^A_B (M^B_\alpha)^{\mathrm{T}}, \qquad \lambda_{\dot{\alpha}A} = -(\lambda_{\dot{\alpha}B})^{\mathrm{T}} R^B_A. \tag{6.7}$$

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After the orbifolding for the one-instanton condition, (1,0,0,0), these relations yield

$$\lambda_{\dot\alpha} = -\lambda_{\dot\alpha}, \qquad\qquad D^c = -D^c. \tag{6.8}$$

Thus, λ and D^c must vanish. It implies that there are no bosonic and fermionic ADHM constraints in $USp(2N_c)$ gauge theory as is well-known.

For the fields in the D3-D(-1) sector, the orientifold projection gives relations between ω and $\bar\omega$, and, μ and $\bar\mu$:

$$\bar\omega = \omega^{\mathrm{T}}\gamma_-^{-1}, \qquad\qquad \bar\mu^A = R^A_B(\mu^B)^{\mathrm{T}}\gamma_-^{-1}. \tag{6.9}$$

These relations become the following relations after the orbifolding:

$$\bar\omega_u = \omega^v J_{vu}, \qquad\qquad \bar\mu_u = \mu^v J_{vu}, \qquad\qquad \bar\mu'_f = \mu'^g \widetilde{J}_{gf}. \tag{6.10}$$

The superpotential is greatly simplified since the ADHM constraints are absent in this case. The action of our interests becomes

$$S = \omega^u_{\dot\alpha} J_{uv} Q^v_f Q^{\dagger f}_w \omega^{\dot\alpha w} + i\mu'^f \widetilde{J}_{fg} Q^{\dagger g}_u \mu^u + 2i\mu'^f \widetilde{J}_{fg}\bar\psi^g_{\dot\alpha u}\omega^{\dot\alpha u}. \tag{6.11}$$

6.1 Multi-fermion F-terms and the moduli space deformation

In this section, we consider the simplest case of the multi-fermion F-term. As we have seen in the case of SU(2), the multi-fermion F-term is related to the deformation of the moduli space. We generalize the analysis to the symplectic case, and show that our multi-fermion F-term reproduces the moduli deformation.

The multi-fermion F-term W is given by the path integral of the action (6.11):

$$W = \int d\{\omega,\mu\} e^{-S}. \tag{6.12}$$

Also we neglect contributions from gauginos. Then, all of μ in the integrand must come from the second term of (6.11), because the ADHM constraints are absent in this case. If $N_c = N_f$, there are no additional contribution of μ', and this integration yields an ADS-type superpotential, as is well known [32]. In the case of $N_f > N_c$, N_c of μ' are supplied by the second term of (6.11), and the rest of μ' are by the third term. Then, W has the $2(N_f - N_c)$ of the fermions $\bar\psi$. We obtain the simplest multi-fermion F-terms in the case of $N_f = N_c + 1$. In this case, we have the following expression up to an overall constant:

$$\begin{aligned}
W &= \int d^2\omega\; e^{-\omega_{\dot\alpha}(JQQ^\dagger)\omega^{\dot\alpha}} \\
&\qquad\qquad \times \varepsilon^{u_1\cdots u_N}\varepsilon^{f_1\cdots f_{N+2}}(JQ^\dagger)_{u_1 f_1}\cdots(JQ^\dagger)_{u_N f_N}(\widetilde{J}\bar\psi\omega)_{f_{N+1}}(\widetilde{J}\bar\psi\omega)_{f_{N+2}} \\
&= \frac{1}{\det(JQQ^\dagger)}\varepsilon^{u_1\cdots u_{2N_c}}\varepsilon^{f_1\cdots f_{2N_f}}(JQ^\dagger)_{u_1 f_1}\cdots(JQ^\dagger)_{u_{2N_c} f_{2N_c}} \\
&\qquad\qquad \times (\widetilde{J}\bar\psi)_{\dot\alpha v f_{N+1}}(\widetilde{J}\bar\psi)^{\dot\alpha}_{w f_{N+2}}\left((QQ^\dagger)^{-1}J\right)^{vw}
\end{aligned} \tag{6.13}$$

where, $N = 2N_c = 2(N_f - 1)$. Using the Bose statistics of the quarks Q, we can rewrite eq. (6.13) in terms of the "mesons" $M_{fg} = J_{uv}Q_f^u Q_g^v$. Then, we obtain the following compact form of the multi-fermion F-term:

$$W = \frac{\bar{C}'(\bar{M})_{ijkl}}{\bar{C}(\bar{M})_{fg}C(M)^{fg}}\widetilde{G}^{ij}_{mn}\bar{D}\bar{M}^{kl}\bar{D}\bar{M}^{mn}, \tag{6.14}$$

$$\widetilde{G}^{ij}_{kl} = Q^{\dagger i}_u \left((QQ^\dagger)^{-2}\right)^u_v Q^v_k \, \delta^j_l, \tag{6.15}$$

where we define C, $\bar{C}$ and $\bar{C}'$ as

$$C(M)^{ij} = \varepsilon^{ijf_1\cdots f_N} M_{f_1f_2}\cdots M_{f_{N-1}f_N}, \tag{6.16}$$

$$\bar{C}(\bar{M})_{ij} = \varepsilon_{ijf_1\cdots f_N}\bar{M}^{f_1f_2}\ldots\bar{M}^{f_{N-1}f_N}, \tag{6.17}$$

and

$$\bar{C}'(\bar{M})_{ijkl} = \varepsilon_{ijklf_1\cdots f_{N-2}}\bar{M}^{f_1f_2}\ldots\bar{M}^{f_{N-3}f_{N-2}}. \tag{6.18}$$

Let us consider the relation between our multi-fermion F-term and the moduli space deformation [32]. We treat only the color symmetry as the gauge symmetry, but the flavor symmetry as a global symmetry by neglecting the flavor gauge field. Then, our model has $USp(2N_c)$ color gauge symmetry and $\mathrm{U}(2N_f)$ flavor global symmetry. The classical moduli space is described by the "mesons" M_{fg}. For $N_f > N_c$, these mesons satisfy the classical constraint

$$\mathrm{Pf}\, M = 0, \tag{6.19}$$

which is a trivial consequence of the Bose statistics of the quark fields Q. In the case of $N_f = N_c + 1$, this constraint is modified to

$$\mathrm{Pf}\, M = \Lambda \tag{6.20}$$

by the quantum effect of instantons.

The quantum moduli space can be converted to the classical moduli space by a non-holomorphic change of variables. Introducing the new coordinates $\widetilde{M} = M - \delta M$, the deformation of the complex structure gives rise to the additional term of the form

$$\omega_{ijkl}G^{ij}_{mn}\bar{D}\bar{M}^{kl}\bar{D}\bar{M}^{mn}, \tag{6.21}$$

where G^{ij}_{kl} is the metric of the moduli space, and ω_{ijkl} is given by

$$\left(\frac{\partial}{\partial\bar{M}^{kl}} + \omega_{ijkl}\frac{\partial}{\partial M_{ij}}\right)\widetilde{M}_{mn} = 0. \tag{6.22}$$

In this case, the moduli deformation is described by

$$\delta M_{ij} = \frac{\bar{C}(\bar{M})_{ij}}{\bar{C}(\bar{M})_{fg}C(M)^{fg}}, \tag{6.23}$$

where we have used the definitions of (6.16) and (6.17) again. Then, we have

$$\omega_{ijkl} = \frac{\bar{C}'(\bar{M})_{ijkl}}{\bar{C}(\bar{M})_{fg}C(M)^{fg}} - \frac{\bar{C}(\bar{M})_{ij}\bar{C}'(\bar{M})_{klmn}C(M)^{mn}}{\left(\bar{C}(\bar{M})_{fg}C(M)^{fg}\right)^2}, \tag{6.24}$$

where C' is defined in (6.18). We do not know the explicit form of the metric G but we can estimate it from the asymptotic form of the Kähler potential K. The Kähler potential for mesons M equals to that of quarks Q at the asymptotic region of the moduli space. Since the D-flatness condition for the symplectic gauge group is

$$J_{uw}Q^w_f Q^{\dagger f}_v = Q^{\dagger f}_u Q^w_f J_{wv}, \tag{6.25}$$

the Kähler potential can be written as

$$K = \mathrm{tr}\, Q^\dagger Q = \mathrm{tr}\,\sqrt{JQ^{\mathrm{T}}Q^{\dagger\,\mathrm{T}}J^{-1}QQ^\dagger}. \tag{6.26}$$

We can read off an expression of the metric from the last expression by using the following relation:

$$\begin{aligned}\frac{\partial^2 K}{\partial Q^{\dagger g}_v \partial Q^u_f} &= \frac{\partial^2 K}{\partial \bar{M}^{kl}\partial M_{ij}}\frac{\partial M_{ij}}{\partial Q^u_f}\frac{\partial \bar{M}^{kl}}{\partial Q^{\dagger g}_v} \\ &= G^{ij}_{kl}\frac{\partial M_{ij}}{\partial Q^u_f}\frac{\partial \bar{M}^{kl}}{\partial Q^{\dagger g}_v}.\end{aligned} \tag{6.27}$$

Thus, we obtain

$$\begin{aligned}G^{ij}_{kl} = \delta^j_k \bar{B}^i_l + \bar{B}^j_k\delta^i_l - \frac{1}{2}(D^j_k\bar{B}^i_l + \bar{B}^j_k D^i_l) \\ + \frac{1}{2}\sum_{r=1}^{\infty}(-1)^r\left[(B^r)^j_k(\bar{B}^{r+1})^i_l + (\bar{B}^{r+1})^j_k(B^r)^i_l\right]\end{aligned} \tag{6.28}$$

where

$$B^i_j = Q^{\dagger i}_u Q^u_j, \qquad \bar{B}^i_j = Q^{\dagger i}_u\left((QQ^\dagger)^{-2}\right)^u_v Q^v_j, \qquad D^i_j = Q^{\dagger i}_u\left((QQ^\dagger)^{-1}\right)^u_v Q^v_j. \tag{6.29}$$

Substituting (6.24) and (6.28) into (6.21), we obtain the multi-fermion F-term generated by one instanton. Using the relation $M_{fg} = J_{uv}Q^u_f Q^v_g$ and the Bose statistics of Q, this multi-fermion F-term is simplified to the following form:

$$W = \frac{\bar{C}'(\bar{M})_{ijkl}}{\bar{C}(\bar{M})_{fg}C(M)^{fg}}\widetilde{G}^{ij}_{mn}\bar{D}\bar{M}^{kl}\bar{D}\bar{M}^{mn}, \tag{6.30}$$

$$\widetilde{G}^{ij}_{kl} = \bar{B}^i_k\delta^j_l, \tag{6.31}$$

which is equivalent to our multi-fermion F-term (6.14).

6.2 Higher multi-fermion F-terms

In the case of $N_f > N_c + 1$, the moduli space is not deformed by effects of instantons, and we cannot relate multi-fermion F-terms to the moduli space deformation. However, multi-fermion F-terms themselves exist even in these cases. These multi-fermion F-terms are also found from the purely field theoretical analysis. Here, we restrict ourselves to the simplest gauge group of $USp(2) \simeq \mathrm{SU}(2)$, and show that our analysis reproduces the result from field theory.

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In general cases of $N_f > N_c$, we have the following expression of the multi-fermion F-term:

$$
\begin{aligned}
W =& \int d^2\omega\ e^{-\omega_{\dot\alpha}(JQQ^\dagger)\omega^{\dot\alpha}} \\
&\times \varepsilon^{u_1\cdots u_{2N_c}}\varepsilon^{f_1\cdots f_{2N_f}}(JQ^\dagger)_{u_1f_1}\cdots(JQ^\dagger)_{u_{2N_c}f_{2N_c}}(\widetilde{J}\bar\psi\omega)_{f_{2N_c+1}}\cdots(\widetilde{J}\bar\psi\omega)_{f_{2N_f}} \\
=& \sum_\sigma \frac{1}{\det(JQQ^\dagger)}\varepsilon^{u_1\cdots u_{2N_c}}\varepsilon^{f_1\cdots f_{2N_f}}(JQ^\dagger)_{u_1f_1}\cdots(JQ^\dagger)_{u_{2N_c}f_{2N_c}} \\
&\times(\widetilde{J}\bar\psi)_{\dot\alpha_1 v_1 f_{2N_c+1}}\cdots(\widetilde{J}\bar\psi)_{\dot\alpha_{N_f-N_c}v_{N_f-N_c}f_{N_f+N_c}}(\widetilde{J}\bar\psi)^{\dot\alpha_1}_{w_1 f_{N_f+N_c+1}}\cdots(\widetilde{J}\bar\psi)^{\dot\alpha_{N_f-N_c}}_{w_{N_f-N_c}f_{2N_f}} \\
&\times\Big((QQ^\dagger)^{-1}J\Big)^{v_1\sigma(w_1)}\cdots\Big((QQ^\dagger)^{-1}J\Big)^{v_{N_f-N_c}\sigma(w_{N_f-N_c})}. \qquad (6.32)
\end{aligned}
$$

where, σ indicates the permutation of indices, which can be removed by using the Fierz identity. Using the relation $M_{fg} = Q^u_f Q^v_g$ and the Bose statistics of quarks Q, we can simplify it into the following compact form:

$$
W = \frac{1}{\det(JQQ^\dagger)}\varepsilon_{f_1\cdots f_{2N_f}}\bar M^{f_1f_2}\ldots\bar M^{f_{2N_c-1}f_{2N_c}}\mathcal{O}^{f_{2N_c+1}f_{2N_c+2}}\ldots\mathcal{O}^{f_{2N_f-1}f_{2N_f}}, \qquad (6.33)
$$

where

$$
\mathcal{O}^{fg} = \bar\psi^f_{\dot\alpha u}\left[(QQ^\dagger)^{-1}\right]^u_w J^{wv}\bar\psi^{\dot\alpha g}_v. \qquad (6.34)
$$

Using the Bose statistics of Q again, and extending $\bar\psi$ to the superfield $\bar D\bar Q$, we can replace $\mathcal{O}$ by $\widetilde{\mathcal{O}}$ which is defined as

$$
\widetilde{\mathcal{O}}^{fg} = \bar D\bar M^{ff'}\left[Q^{\mathrm{T}}J(QQ^\dagger)^{-1}\right]_{f'g'}\bar D\bar M^{g'g}. \qquad (6.35)
$$

In the case of $USp(2) = \mathrm{SU}(2)$, D-flatness condition gives the following relation:

$$
(QQ^\dagger)^u_v = \delta^u_v\,\mathrm{tr}(QQ^\dagger) = \delta^u_v\sqrt{\mathrm{tr}\,\bar MM}. \qquad (6.36)
$$

Then, we obtain

$$
\widetilde{\mathcal{O}}^{fg} = (\mathrm{tr}\,\bar MM)^{-3/2}M_{f'g'}\bar D\bar M^{ff'}\bar D\bar M^{gg'}, \qquad (6.37)
$$

Using this expression, we can rewrite the multi-fermion F-term as

$$
\begin{aligned}
W &= \big(\mathrm{tr}\,\bar MM\big)^{-(3n-1)/2}\varepsilon_{f_1\cdots f_{2N_f}}\bar M^{f_1f_2}\bar{\mathcal{O}}^{f_3f_4}\ldots\bar{\mathcal{O}}^{f_{2N_f-1}f_{2N_f}}, \qquad (6.38)\\
\bar{\mathcal{O}}^{ij} &= M_{kl}\bar D_{\dot\alpha}\bar M^{ik}\bar D^{\dot\alpha}\bar M^{lj}. \qquad (6.39)
\end{aligned}
$$

Therefore, this multi-fermion F-term is equivalent to that in [1].

7. Conclusions and discussions

In this paper, we have investigated the instanton in $\mathcal{N} = 1$ SQCD by using the D-brane effective theory. In SQCD with gauge group $\mathrm{SU}(N_c)$ and $N_f = N_c$ flavors, instantons modify the moduli space of vacua. This effect can be described by the multi-fermion F-terms, and we have derived these terms from the D-brane effective action. SQCD can be obtained by introducing the orbifolding into the D3-brane effective theory, and the gauge instanton corresponds to the D-instanton on the D3-brane. The effective potential generated by instantons can be obtained from the D-instanton effective action. The multi-fermion F-terms are given in terms of integral with respect to the instanton moduli. This integration generally gives complicated expression. We have considered the simplest case of SU(2) gauge group, for which the potential was calculated in [1] by using purely field theoretical techniques. Our result correctly reproduced that in [1]. We have also considered the case of symplectic gauge group. In this case, we have obtained much simpler results than those in the case of the unitary group. This is due to the fact that the ADHM constraints are absent for the symplectic gauge group. We have shown that the deformation of the moduli space is described by the multi-fermion F-term derived from D-instantons, for $N_f = N_c + 1$. We have also calculated multi-fermion F-terms with more fermions, which appear for the theory with more flavors. For $USp(2) \sim \mathrm{SU}(2)$, our result agrees with that in [1], again.

We would like to comment on the case of the orthogonal group. SQCD with this gauge group can be obtained by introducing the orientifold which is opposite to that for the symplectic group; symmetric projection is imposed on the Chan-Paton factor of D3-branes and the anti-symmetric projection on that of D-instantons. We can obtain multi-fermion F-terms similar to those in the case of unitary gauge group for $N_f = N_c$ or symplectic gauge group for $N_f = N_c + 1$, i.e. those in the form of (3.15). Due to the anti-symmetric projection, the size of the matrices for D-instantons must be even, and we should take $k = 2$ for one-instanton. Then, the Chan-Paton factor for D-instantons becomes $USp(2) \sim \mathrm{SU}(2)$, and consequently, there are three sets of the ADHM constraints corresponding to three SU(2) generators. The fermionic ADHM constraints supply six fermions μ. Since there are $2N_c$ fermions μ and $2N_f$ fermions μ', we obtain the multi-fermion F-terms in the form of (3.15), if $N_f = N_c - 2$. However, there are no constraint on the moduli space for $N_f = N_c - 2$, and therefore, we cannot see any relation between the multi-fermion F-terms and the deformation of the moduli space [33]. Furthermore, a large number of the ADHM constraint makes the integration in the expression of multi-fermion F-terms much more complicated.

It would be interesting to study the relation between multi-fermion F-terms and the deformation of the moduli space in the case of $\mathrm{SU}(N_c)$ for $N_c > 2$. In order to study this relation, we have to obtain an explicit form of the metric on the moduli space. Even in the case of SU(2) and symplectic group, the metric is determined in the asymptotic region by using symmetries. It is interesting to describe deformation of the moduli space in full detail. These issues are left for future studies.

Another interesting problem is a generalization to other models. In this paper, we have demonstrated the D-instanton derivation of multi-fermion F-terms which are related to the deformation of the moduli space. This method can be applied to other models and will show how the moduli space is deformed by instantons. It would also be interesting to study stringy effects of multi-fermion F-terms.

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A. Supersymmetry transformations

In this appendix, we describe the supersymmetry transformation of the effective theory. The D3-brane effective theory is the $\mathcal{N} = 4$ super Yang-Mills theory. The followings are their transformations:

$$
\begin{aligned}
\delta A_\mu &= i\bar{\xi}_{\dot{\alpha}A}\bar{\sigma}^{\mu\dot{\alpha}\beta}\Psi^A_\beta + \xi^{\alpha A}\sigma^\mu_{\alpha\dot{\beta}}\bar{\Psi}^{\dot{\beta}}_A, && \text{(A.1)}\\
\delta\Psi^{\alpha A} &= \frac{i}{2}\sigma^{\mu\nu\alpha}_\beta\xi^{\beta A}F_{\mu\nu} + \frac{i}{2}\varepsilon^{ABCD}\bar{\xi}_{\dot{\beta}B}\bar{\sigma}^{\mu\dot{\beta}\alpha}D_\mu\Phi_{AB} && \text{(A.2)}\\
&\qquad + + \left(\frac{1}{8}\xi^{\alpha A}\varepsilon^{BCDE} - \frac{1}{2}\varepsilon^{ABCD}\xi^{\alpha E}\right)\Phi_{BC}\Phi_{DE},\\
\delta\bar{\Psi}_{\dot{\alpha}A} &= \frac{i}{2}\bar{\xi}_{\dot{\beta}A}\bar{\sigma}^{\mu\nu\dot{\beta}}_{\dot{\alpha}}F_{\mu\nu} + i\xi^B_\beta\sigma^\mu_{\beta\dot{\alpha}}D_\mu\Phi_{AB} && \text{(A.3)}\\
&\qquad + \varepsilon^{BCDE}\left(\frac{1}{8}\bar{\xi}_{\dot{\alpha}A}\Phi_{BC}\Phi_{DE} - \frac{1}{2}\bar{\xi}_{\dot{\alpha}E}\Phi_{AB}\Phi_{CD}\right),\\
\delta\Phi_{AB} &= 2i\varepsilon_{ABCD}\xi^C_\alpha\Psi^{\alpha D} - 2i\left(\bar{\xi}_{\dot{\alpha}A}\bar{\Psi}^{\dot{\alpha}}_B - \bar{\xi}_{\dot{\alpha}B}\bar{\Psi}^{\dot{\alpha}}_A\right). && \text{(A.4)}
\end{aligned}
$$

where, we rewrote the R-symmetry in terms of SU(4) using the definition, $\Phi_{AB} = \bar{\Sigma}^a_{AB}\Phi^a$.

Introducing the D-instanton, a half of the supersymmetry is broken and the unbroken symmetry is generated by supercharges $\bar{Q}^A_{\dot{\alpha}}$. We list these supersymmetry transformations of the fields on the D-instanton. The fields in the D(-1)-D(-1) sector are given by dimensional reduction of the $\mathcal{N} = 4$ super Yang-Mills theory. Their transformations are

$$
\begin{aligned}
\delta a_\mu &= \frac{i}{2}\bar{\xi}_{\dot{\alpha}A}\bar{\sigma}^{\mu\dot{\alpha}\beta}M^A_\beta, && \text{(A.5)}\\
\delta\chi^a &= -i\bar{\xi}_{\dot{\alpha}A}\Sigma^{aAB}\lambda^{\dot{\alpha}}_B, && \text{(A.6)}\\
\delta M^{\alpha A} &= -\frac{1}{2}\bar{\xi}_{\dot{\beta}B}\bar{\sigma}^{\mu\dot{\beta}\alpha}\Sigma^{aAB}[\chi^a, a_\mu], && \text{(A.7)}\\
\delta\lambda_{\dot{\alpha}A} &= \frac{1}{2}\bar{\xi}_{\dot{\alpha}B}\bar{\Sigma}^{ab\,B}_A[\chi^a,\chi^b] + \bar{\xi}_{\dot{\beta}A}\bar{\sigma}^{\mu\nu\dot{\beta}}_{\dot{\alpha}}[a_\mu, a_\nu]. && \text{(A.8)}
\end{aligned}
$$

By introducing the auxiliary fields D^c, the transformation of $\lambda_{\dot{\alpha}A}$ becomes

$$
\delta\lambda_{\dot{\alpha}A} = \frac{1}{2}\bar{\xi}_{\dot{\beta}A}\tau^{c\dot{\beta}}_{\dot{\alpha}}D^c. \tag{A.9}
$$

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And the transformations in the D3-D(-1) sector are the followings:

$$\delta\omega_{\dot\alpha} = -i\bar\xi_{\dot\alpha A}\mu^A, \tag{A.10}$$

$$\delta\bar\omega_{\dot\alpha} = i\bar\xi_{\dot\alpha A}\bar\mu^A, \tag{A.11}$$

$$\delta\mu^A = \varepsilon^{ABCD}\bar\xi_{\dot\alpha B}\left(\omega^{\dot\alpha}\chi_{CD} + \Phi_{CD}\omega^{\dot\alpha}\right), \tag{A.12}$$

$$\delta\bar\mu^A = \varepsilon^{ABCD}\bar\xi_{\dot\alpha B}\left(\chi_{CD}\bar\omega^{\dot\alpha} + \bar\omega^{\dot\alpha}\Phi_{CD}\right), \tag{A.13}$$

where, $\chi_{AB} = \bar\Sigma^a_{AB}\chi^a$.

B. Multi-fermion F-terms for general SU(N)

In this appendix, we show an explicit expression of the multi-fermion F-term for general $N_f \geq N_c$ in the case of unitary gauge group. The expression is quite complicated if all fields in the D3-D(-1) sector are integrated out. Therefore, we do not perform all of the integration. The multi-fermion F-term can be expressed in terms of the integral with respect to ω:

$$\begin{aligned}
W = &\int d\omega^2 d\bar\omega^2\ \delta^{(3)}(\bar\omega_{\dot\alpha u}(\tau^c)^{\dot\alpha}_{\dot\beta}\omega^{\dot\beta u})\, e^{-\frac{1}{2}\bar\omega_{\dot\alpha u}\left(Q^u_f Q^{\dagger f}_v + \widetilde{Q}^{\dagger u}_f \widetilde{Q}^f_v\right)\omega^{\dot\alpha v}}\varepsilon^{u_1\cdots u_{N_c}}\varepsilon_{v_1\cdots v_{N_c}}\varepsilon_{f_1\cdots f_{N_f}}\varepsilon^{g_1\cdots g_{N_f}}\\
&\times\Bigg[\frac{1}{N_c!(N_c-2)!(N_f-N_c)!(N_f-N_c+2)!}\\
&\quad\times Q^{\dagger f_1}_{u_1}\cdots Q^{\dagger f_{N_c-2}}_{u_{N_c-2}}\widetilde{Q}^{\dagger v_1}_{g_1}\cdots\widetilde{Q}^{\dagger v_{N_c}}_{g_{N_c}}\\
&\quad\times \bar\omega_{\dot\gamma u_{N_c-1}}\bar\omega^{\dot\gamma}_{u_{N_c}}\bar\psi^{f_{N_c-1}}_{\dot\alpha_1 w_1}\omega^{\dot\alpha_1 w_1}\cdots\bar\psi^{f_{N_f}}_{\dot\alpha_{N_f-N_c+2}w_{N_f-N_c+2}}\omega^{\dot\alpha_{N_f-N_c+2}w_{N_f-N_c+2}}\\
&\quad\times \bar\omega_{\dot\beta_1 y_1}\bar{\tilde\psi}^{\dot\beta y_1}_{g_{N_c+1}}\cdots\bar\omega_{\dot\beta_{N_f-N_c}y_{N_f-N_c}}\bar{\tilde\psi}^{\dot\beta_{N_f-N_c}y_{N_f-N_c}}_{g_{N_f}}\\
&\quad+\frac{2}{[(N_c-1)!]^2[(N_f-N_c+1)!]^2}\\
&\quad\times Q^{\dagger f_1}_{u_1}\cdots Q^{\dagger f_{N_c-1}}_{u_{N_c-1}}\widetilde{Q}^{\dagger v_1}_{g_1}\cdots\widetilde{Q}^{\dagger v_{N_c-1}}_{g_{N_c-1}}\\
&\quad\times \bar\omega_{\dot\gamma u_{N_c}}\omega^{\dot\gamma v_{N_c}}\bar\psi^{f_{N_c}}_{\dot\alpha_1 w_1}\omega^{\dot\alpha_1 w_1}\cdots\bar\psi^{f_{N_f}}_{\dot\alpha_{N_f-N_c+1}w_{N_f-N_c+1}}\omega^{\dot\alpha_{N_f-N_c+1}w_{N_f-N_c+1}}\\
&\quad\times \bar\omega_{\dot\beta_1 y_1}\bar{\tilde\psi}^{\dot\beta y_1}_{g_{N_c}}\cdots\bar\omega_{\dot\beta_{N_f-N_c+1}y_{N_f-N_c+1}}\bar{\tilde\psi}^{\dot\beta_{N_f-N_c+1}y_{N_f-N_c+1}}_{g_{N_f}}\\
&\quad+\frac{1}{N_c!(N_c-2)!(N_f-N_c)!(N_f-N_c+2)!}\\
&\quad\times Q^{\dagger f_1}_{u_1}\cdots Q^{\dagger f_{N_c}}_{u_{N_c}}\widetilde{Q}^{\dagger v_1}_{g_1}\cdots\widetilde{Q}^{\dagger v_{N_c-2}}_{g_{N_c-2}}\\
&\quad\times \omega^{v_{N_c-1}}_{\dot\gamma}\omega^{\dot\gamma v_{N_c}}\bar\psi^{f_{N_c+1}}_{\dot\alpha_1 w_1}\omega^{\dot\alpha_1 w_1}\cdots\bar\psi^{f_{N_f}}_{\dot\alpha_{N_f-N_c}w_{N_f-N_c}}\omega^{\dot\alpha_{N_f-N_c}w_{N_f-N_c}}\\
&\quad\times \bar\omega_{\dot\beta_1 y_1}\bar{\tilde\psi}^{\dot\beta y_1}_{g_{N_c-1}}\cdots\bar\omega_{\dot\beta_{N_f-N_c+2}y_{N_f-N_c+2}}\bar{\tilde\psi}^{\dot\beta_{N_f-N_c+2}y_{N_f-N_c+2}}_{g_{N_f}}\Bigg].
\end{aligned}\tag{B.1}$$

The fields on the D-instanton correspond to the moduli of the instanton, and ω's are the gauge direction and the size of the instanton. For example, the instanton solution of the

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gauge field A_μ is written in terms of ω as

$$A_\mu(x) = \omega^u_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\dot\beta}\bar\omega^{\dot\beta}_v \frac{(x-x_0)^\nu}{(x-x_0)^2[(x-x_0)^2+\frac{1}{2}\bar\omega_{\dot\gamma w}\omega^{\dot\gamma w}]}, \tag{B.2}$$

where, we have taken the singular gauge, and $x_0^\mu = a^\mu$ is the position of the instanton.

C. Constrained Gaussian integral

In this appendix, we describe the calculation of the constraint Gaussian integral. Let us define F by

$$F = \int d\omega\, d\bar\omega\ \delta^{(3)}(\bar\omega\tau^c\omega)\, e^{-\bar\omega_{\dot\alpha}A\omega^{\dot\alpha}} \tag{C.1}$$

where A is an $N\times N$ matrix. Writing the δ-function in terms of the contour integral, we obtain

$$F = \int d\omega\, d\bar\omega\, d^3k\ e^{-\bar\omega_{\dot\alpha}A\omega^{\dot\alpha}+ik_a\bar\omega_{\dot\alpha}\tau^{a\dot\alpha}_{\dot\beta}\omega^{\dot\beta}} = \int d\Omega\, d\bar\Omega\, d^3k\ e^{-\bar\Omega\tilde A\Omega} \tag{C.2}$$

where $\tilde A$ is the $2N\times 2N$ matrix which is defined as

$$\tilde A = \begin{pmatrix} A-ik_3 & -k_1-k_2 \\ -ik_1+k_2 & A+ik_3 \end{pmatrix}, \tag{C.3}$$

while Ω and $\bar\Omega$ are defined as

$$\Omega = \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}, \qquad\qquad \bar\Omega = \begin{pmatrix}\bar\omega_1,\ \bar\omega_2\end{pmatrix}. \tag{C.4}$$

Then integrating over Ω and $\bar\Omega$, we have

$$F = \int d^3k\frac{1}{\det\tilde A} = \int dk\frac{1}{\det(A^2+k^2)} \tag{C.5}$$

where we used the following formula:

$$\det\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \det(AD-BC) \tag{C.6}$$

if A, B, C, D mutually commute. Diagonalizing A into $diag(a_1, a_2\cdots a_N)$ and picking up all residues in the upper half plane of k, we obtain

$$F = \int dk\frac{k^2}{\prod_{m=1}^N(a_m^2+k^2)} = \sum_{n=1}^N\frac{a_n}{\prod_{m\neq n}(a_m^2-a_n^2)} \tag{C.7}$$

This can be written in terms of A as

$$F = \mathrm{tr}\,\frac{A}{\Sigma_{r=0}^{N-1}A^{2r}C^{(r)}} \tag{C.8}$$

where

$$C^{(r)\,i}_{\ \ \ \,j} = \frac{1}{(N-r-1)!r!}\varepsilon^{in_1\cdots n_r\, l_1\cdots l_{N-(r+1)}}\varepsilon_{jn_1\cdots n_r\, k_1\cdots k_{N-(r+1)}}(A^2)^{k_1}_{l_1}\cdots(A^2)^{k_{N-(r+1)}}_{l_{N-(r+1)}} \tag{C.9}$$

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